Lecture 9

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DIRAC GEOMETRY

references: M. Gualtieri's thesis, chapter 1

H. Bursztyn, A Brief Introduction to Dirac Geometry

C. Zapata's thesis, sections 2.5.3, 4.2

Definition Proposition Exercise

(I) COURANT SPACES

A Courant space is a triple $(C, \langle , \rangle, \rho)$ with $(C, \langle , \rangle)$ an inner product space and $\rho : C \to V$ a linear map called the anchor. The inner product induces the usual notions of orthogonality and we indentify subspaces:

$$N \subset C \quad : \quad \begin{array}{ll} N \subset N^\perp & \text{isotropic} \\ N^\perp \subset N & \text{coisotropic} \\ N = N^\perp & \text{Lagrangian} \end{array}$$

The musical isomorphism induced by $\langle , \rangle$, $C \underset{\#}{\overset{\flat}{\rightleftarrows}} C^*$ gives the complex

$$V^* \xrightarrow{\# \circ \rho^*} C \xrightarrow{\rho} V$$

When this becomes a short exact sequence $(C, \langle , \rangle, \rho)$ is called an exact Courant space, note that in this case

All Courant spaces will

$\ker\rho = \#(\rho^*(V^*)) \subset (C,\langle,\rangle)$ is maximally isotropic. be assumed to be exact.

Opposite Courant space $(\overline{C},\overline{\langle,\rangle},\overline{\rho}) := (C,-\langle,\rangle,\rho)$

Direct Sum $\quad (C\oplus C', \langle,\rangle\oplus\langle,\rangle', \rho\oplus\rho')$

Standard Courant space $\mathbb{V} := (V\oplus V^*, \langle,\rangle, pr_1)$

$$\langle v\oplus\alpha, w\oplus\beta\rangle := \frac{1}{2}(\beta(v)+\alpha(w)).$$

A Dirac space is a maximally isotropic subspace $D\subset C$ for which there exist $W\subset V$ and $\tilde{W}\subset V^*$ such that we have the short exact sequence:

$$0 \longrightarrow \tilde{W} \xrightarrow{\#\rho^*} D \xrightarrow{\rho} W \longrightarrow 0$$

The space $W=\rho(D)$ is usually called the range of D.

Proposition 9.1 A Dirac space $D\subset(C,\langle,\rangle,\rho)$ defines a 2-form ω_D on its range $\rho(D)\subset V$.

proof. define Dirac extensions of elements in $W=\rho(D)\subset V$ as.

$w_1,w_2\in W$ extend to $a_1,a_2\in C$ s.t. $\rho(a_1)=w_1$, $\rho(a_2)=w_2$.

$\epsilon_1,\epsilon_2\in\tilde{W}$ s.t. $a_1+\#\rho^*(\epsilon_1)\in D$, $a_2+\#\rho^*(\epsilon_2)\in D$

define:

$$\omega_D(w_1,w_2) := \langle a_1, \#\rho^*(\epsilon_2)\rangle = \epsilon_2(w_1)$$

$$\shortparallel$$

$$-\langle\#\rho^*(\epsilon_1), a_2\rangle = -\epsilon_1(w_2).$$

Isotropic relation is an isotropic subspace

$\Lambda: A \dashrightarrow B \qquad \Lambda\subset B\oplus\overline{A}$

$(A,\langle,\rangle_A,\alpha:A\to V)\quad(B,\langle,\rangle_B,\beta:B\to W)$

we say $a \in A$ and $b \in B$ are Λ-related if $b \oplus a \in \Lambda$, in this case it is obvious to check:

$$a_1 \sim_\Lambda b_1 \,, \quad a_2 \sim_\Lambda b_b \quad \Rightarrow \quad \langle a_1, a_2 \rangle_A = \langle b_1, b_2 \rangle_B \,.$$

An isotropic relation $\Gamma : A \dashrightarrow B$ is called a Courant morphism if there exists a map $\gamma : V \to W$ such that

$$b \oplus a \in \Gamma \Rightarrow \beta(b) = \gamma(\alpha(a)) \,.$$

this makes Γ_γ into a Dirac space of the direct sum:

$$0 \to \mathrm{graph}(\psi^*) \to \Gamma_\gamma \to \mathrm{graph}(\psi) \to 0 \,.$$

More specifically, a Courant map is defined as a linear map $\Psi : A \to B$ such that

$$\exists \psi : V \to W \text{ s.t. } \beta \circ \Psi = \psi \circ \alpha \quad \text{and} \quad \Psi^* \langle , \rangle_B = \langle , \rangle_A \,.$$

Show

$\mathrm{graph}(\Psi) \subset B \oplus \bar{A}$ Courant morphism $\Leftrightarrow$ Ψ Courant map.

Defining composition as relations we easily check

$$\Gamma_\gamma : A \dashrightarrow B \,, \quad \Lambda_\lambda : B \dashrightarrow C$$

$$\Lambda_\lambda \circ \Gamma_\gamma = (\Lambda \circ \Gamma)_{\lambda \circ \gamma} : A \dashrightarrow C$$

This defines the category of Courant spaces Crnt.

The presence of 2-forms associated to Dirac spaces gives the following fibration sequence:

$$\Lambda^2 V^* \to \mathrm{Crnt}(A,B) \to \mathrm{Vect}(V,W)$$

In the case of Courant isomorphisms, the above sequence gives an abelian extension sequence for the group of automorphisms

$$0 \to (\Lambda^2 V^*, +) \to \mathrm{Aut}_{\mathrm{Crnt}}(A) \to GL(V) \to 0 .$$

An isotropic splitting is an splitting of the short exact sequence of a Courant space

$$\begin{tikzcd} 0 \arrow[r] & V^* \arrow[r] & C \arrow[r, shift left, "\rho"] & V \arrow[l, shift left, "\nabla"] \arrow[r] & 0 \end{tikzcd}$$

whose image $\nabla(V) \subset (C, <,>)$ is isotropic.

Show that two isotropic splittings differ by a 2-form on V.

Proposition 9.2 A choice of isotropic splitting for $(C, <,>, \rho: C \to V)$ induces an isomorphism of Courant spaces

$$(C, <,>, \rho: C \to V) \cong (\mathbb{V} := V \oplus V^*, <,>, pr_1).$$

proof.

Courant morphisms for standard Courant spaces have a much more explicit expression:

$$\mathrm{Crnt}(\mathbb{V}, \mathbb{W}) = \Lambda^2 V^* \times \mathrm{Vect}(V, W)$$

with $(B, \varphi): \mathbb{V} \dashrightarrow \mathbb{W}$ given by:

$$v \oplus \alpha \sim_\psi^B w \oplus \beta \iff \psi(v) = w \quad \& \quad \alpha + i_v B = \psi^* \beta .$$

an composition

$$(\phi, B') \circ (\psi, B) = (B + \psi^* B', \phi \circ \psi) .$$

Standard Courant automorphisms are now given by the semidirect product of groups:

$$\mathrm{Aut}_{\mathrm{Crnt}}(\mathbb{V}) = \Lambda^2 V^* \rtimes GL(V) .$$

II DORFMAN ALGEBRAS

Leibniz algebra $(\mathfrak{a}, [\,,\,])$ with $[\,,\,]$ $\mathbb{R}$-bilinear bracket:

$$\forall a,b,c \in \mathfrak{a} \qquad [a,[b,c]] = [[a,b],c] + [b,[a,c]].$$

Leibniz morphism $\Psi : \mathfrak{a} \to \mathfrak{b}$, $\Psi([a,b]_{\mathfrak{a}}) = [\Psi(a), \Psi(b)]_{\mathfrak{b}}$.

Note that the left and right adjoint maps are not equivalent.

Leibniz representation is a pair of maps

$\mathfrak{h}$ vector space, $\quad [\,,\,]_L : \mathfrak{a} \times \mathfrak{h} \to \mathfrak{h}$, $\quad [\,,\,]_R : \mathfrak{h} \times \mathfrak{a} \to \mathfrak{h}$

such that all the possible Leibniz identities hold e.g.:

$$a,b \in \mathfrak{a},\ h \in \mathfrak{h} \qquad [h,[a,b]]_R = [[h,a]_R, b]_R + [a,[h,b]_R]_L$$

$$[a,[b,h]_L]_L = \cdots \quad \text{etc.}$$

We can define the cochain complex:

$$C^k(\mathfrak{a}, \mathfrak{h}) := \otimes^k \mathfrak{a}^* \otimes \mathfrak{h}$$

and define a differential

$$\delta^0 : C^0(\mathfrak{a}, \mathfrak{h}) \to C^1(\mathfrak{a}, \mathfrak{h})$$

$$h \longmapsto [-,h]_L - [h,-]_R$$

$$\delta^1 : C^1(a,h) \longrightarrow C^2(a,h)$$
$$\eta \longmapsto \delta^1\eta$$

$$\delta^1\eta(a,b) := [a,\eta(b)]_L + [\eta(a),b]_R - \eta([a,b])$$

and check:

$$\delta^1\delta^0 h\,(a,b) = [a,[b,h]_L]_L - [a,[h,b]_R]_L -$$
$$+ [[a,h]_L,b]_R - [[h,a]_R,b]_R$$
$$- [[a,b],h]_L + [h,[a,b]]_R$$
$$= 0.$$

Extending to higher $\otimes$ powers we obtain the Leibniz cohomology $H^\bullet(a,h)$ as the cohomology of $(C^\bullet(a,h),\delta)$.

A Dorfman algebra a over a Lie algebra $\mathfrak{g}$ is a Leibniz algebra $(a,[,])$ together with a morphism of Leibniz algebras

$$\rho : (a,[,]) \longrightarrow (\mathfrak{g},[,]).$$

By construction, $\rho(a) \subset \mathfrak{g}$ is a Lie sub algebra and $h := \ker\rho$ is a two-sided Leibniz ideal.

A Dorfman algebra is exact when the sequence:

$$0 \longrightarrow h \longrightarrow a \xrightarrow{\rho} \mathfrak{g} \longrightarrow 0$$

is exact and h is an abelian sub algebra. $[h,h]=0$.

A first example of a Dorfman algebra is given by a conventional Lie algebra representation on a vector space $R: \mathfrak{g} \times \mathfrak{h} \to \mathfrak{h}$, the hemisemidirect product Leibniz algebra:

$$\mathfrak{a} := \mathfrak{h} \oplus \mathfrak{g} \qquad [h_1 \oplus g_1, h_2 \oplus g_2] := R_{g_1}(h_2) \oplus [g_1, g_2]$$

$$\rho = pr_2$$

Show $\psi: \mathfrak{g} \times \mathfrak{h} \to \mathfrak{h}$ is Lie rep. $\Longleftrightarrow$ $\mathfrak{h} \oplus \mathfrak{g}$ is Leibniz.

A Dorfman morphism is a Leibniz morphism $\Psi: \mathfrak{a} \to \mathfrak{a}'$ such that $\exists$ Lie morphism $\psi: \mathfrak{g} \to \mathfrak{g}'$ satisfying:

$$\rho' \circ \Psi = \psi \circ \rho \, .$$

Exact Dorfman algebras carry Leibniz representations of $\mathfrak{g}$ on the kernel of the anchor $\ker \rho =: \mathfrak{h}$, we define it by:

$$\begin{aligned} [\,,\,]_L : \mathfrak{g} \times \mathfrak{h} &\to \mathfrak{h} \\ (g, h) &\mapsto [a, h] \end{aligned} \qquad \text{for some } a \in \mathfrak{a} \text{ s.t. } \rho(a) = g$$

Similarly for $[\,,\,]_R$.

Show $[\,,\,]_{L,R}$ are well-defined maps giving a Leibniz representation.

By considering splittings at the level of vector spaces

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{h} \arrow[r] & \mathfrak{a} \arrow[r, shift left, "\rho"] & \mathfrak{g} \arrow[l, shift left, "\nabla"] \arrow[r] & 0 \end{tikzcd}$$

We can define the Leibniz curvature of ∇:

$C_{\bar\nabla} \in C^2(\mathfrak{g};\mathfrak{h})$, $C_{\bar\nabla}(g,g') := [\bar\nabla(g),\bar\nabla(g')]_{\mathfrak{a}} - \bar\nabla([g,g']_{\mathfrak{g}})$

which can be shown to be a cocycle:

$$\delta C_{\bar\nabla} = 0.$$

and choosing a different splitting $\nabla' = \nabla + \eta$, $\eta : \mathfrak{g} \to \mathfrak{h}$, we find:

$$C_{\bar\nabla'} = C_{\bar\nabla} + \delta\eta.$$

We thus see that an exact Dorfman algebra carries an element of the Leibniz 2-cohomology defined as:

$$H := [C_{\bar\nabla}] = [C_{\bar\nabla'}] \in H^2(\mathfrak{g};\mathfrak{h}).$$

This is called the characteristic class of the exact Dorfman algebra.

Proposition 9.3

Two exact Dorfman algebras over the same Lie algebra are isomorphic iff their characteristic classes agree.

proof. (Prop. 2.5.7 in my thesis)

Conversely, we can construct Dorfman algebras from Leibniz representations of Lie algebras:

$[\,,\,]_{L/R} : \mathfrak{g} \times \mathfrak{h} \to \mathfrak{h}$ Leibniz representation of $(\mathfrak{g},[\,,\,])$ Lie algebra.

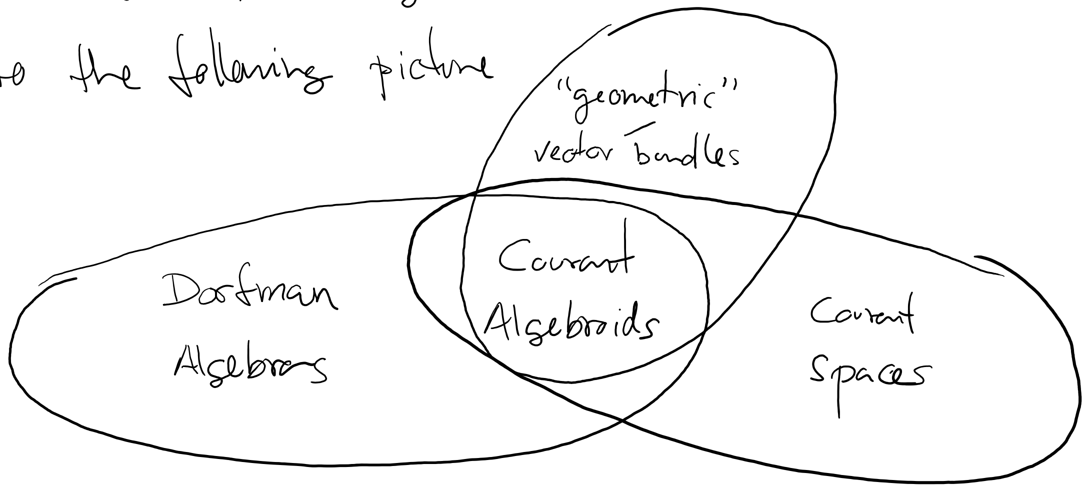
choose a class $[\eta] \in H^2(\mathfrak{g};\mathfrak{h})$ then we define the semidirect product Dorfman algebra as:

$\mathfrak{a} = \mathfrak{h} \oplus \mathfrak{g}$, $[h_1 \oplus g_1, h_2 \oplus g_2] := [g_1, h_2]_L + [h_1, g_2] + \eta(g_1, g_2) \oplus [g_1, g_2]$

$\rho = pr_2$

III) COURANT ALGEBROIDS

In essence, the general definition of Courant algebroid responds to the following picture



Let us motivate the conventional definition by finding a Dorfman algebra structure associated to any Lie algebroid $(A, \rho, [\,,\,])$ and Cartan calculus $(\Gamma(A^*), i_a, \mathcal{L}_a, d)$ (note that we could think of $A = TM$ and ordinary Cartan calculus)

Show that

$$[\,,\,]_L : \Gamma(A) \times \Gamma(A^*) \longrightarrow \Gamma(A^*)$$
$$(a, \alpha) \longmapsto \mathcal{L}_a \alpha$$

$$[\,,\,]_R : \Gamma(A^*) \times \Gamma(A) \longrightarrow \Gamma(A^*)$$
$$(\alpha, a) \longmapsto i_a d\alpha$$

form a Leibniz representation of $(\Gamma(A), [\,,\,])$ on $\Gamma(A^*)$.

Then we consider the semidirect product Dorfman bracket:

$$[a\oplus\alpha, b\oplus\beta] := [a,b]\oplus \mathcal{L}_a\beta - i_b d\alpha$$

(where we have chosen the zero characteristic class in Leibniz cohomology).

From our discussion on (II) this endows $\Gamma(A)\oplus\Gamma(A^*)$ with a Dorfman algebra structure.

The bracket indeed fails to be skewsymmetric, however the failure to do so can be shown to be given by:

$$[a\oplus\alpha, a\oplus\alpha] = 0\oplus d(\alpha(a))$$

So we see that the natural pairing between $\Gamma(A)$ and $\Gamma(A^*)$ controls the skewsymmetry of the bracket. We thus define:

a Courant algebroid as a tuple $\left(\begin{smallmatrix}E\\ \downarrow\\ M\end{smallmatrix}, \langle,\rangle, \rho, [,]\right)$

where $\langle,\rangle\in\Gamma(\odot^2E^*)$ is a symmetric bilinear form,

$\rho_*:(\Gamma(E),[,])\to(\Gamma(TM),[,])$ is a Dorfman algebra and the compatibility conditions:

i) $[a, f\cdot b] = f\cdot[a,b] + \rho_*(a)[f]\cdot b$

ii) $\rho_*(a)[\langle b,c\rangle] = \langle [a,b], c\rangle + \langle b, [a,c]\rangle$

iii) $[a,a] = D\langle a,a\rangle$, $D := \#\circ\rho^*\circ d : C^\infty(M)\to\Gamma(E)$

It follows from this definition that Courant algebroids are bundles of Courant spaces

$$x\in M \qquad \rho_x:(E_x,\langle,\rangle_x)\to T_xM$$

whose sections carry a Dorfman algebra structure.

A Courant algebroid is called exact when each E_x is so, this makes $\Gamma(E)$ into an exact Dorfman algebra:

$$0 \to \Omega^1(M) \xrightarrow{i} \Gamma(E) \xrightarrow{\rho_*} \mathfrak{X}(M) \to 0$$

with $[i(\Omega^1(M)), i\Omega^1(M)] = 0$.

All the notions introduced in I and II for exact Courant spaces and exact Dorfman algebras apply to exact Courant algebroids with the necessary modifications to fit the setting of vector bundles.

The characteristic class of an exact Courant algebroid is an element of the Leibniz cohomology:

$$[\gamma] \in H^2(\mathfrak{X}(M); \Omega^1(M))$$

it is easy to check that this is equivalent to a 3-form H since

$$H^\bullet(\mathfrak{X}(M); \Omega^1(M)) \cong H^{\bullet+1}(M)$$

the de Rham cohomology class $[H] \in H^3(M)$ is called the Severa class of the exact Courant algebroid.

Introducing an isotropic splitting we recover the notion of standard Courant algebroid

$$E \cong TM \oplus T^*M =: \mathbb{T}M$$

with Dorfman bracket given by:

$$[X \oplus \alpha, Y \oplus \beta] = [X, Y] \oplus \mathcal{L}_X \beta - i_Y d\alpha + i_X i_Y H.$$

A Dirac structure (with support on $Q \subset M$) is a subbundle $D \subset E|_Q$ such that

i) $D_q \subset E_q$ is a Dirac space in $\rho_q : (E_q, \langle , \rangle_q) \longrightarrow T_q Q$

ii) $a, b \in \Gamma(E)$, $a|_Q, b|_Q \in \Gamma(D) \Rightarrow [a,b]|_Q \in \Gamma(D)$.

It follows from the above that $\rho(D) \subset TQ$ is an involutive distribution, a leaf $\mathcal{O}$ of this distribution inherits a 2-form ω_D (from the point-wise 2-form ω_{D_q} defined by the dirac space D_q) and it can be shown that

$$d\omega_D(X, Y, Z) = \langle [x,y], z \rangle|_D$$

for $x, y, z \in \Gamma(E)$ such that $\rho_*(x) = X$, $\rho_*(y) = Y$, $\rho_*(z) = Z$.

However, since D is isotropic $\langle , \rangle|_D = 0$ and thus we see that $d\omega_D = 0$, giving a foliation of Q by presymplectic leaves. This is called the Dirac foliation of D.

Courant morphisms are defined from the corresponding linear counterparts fibre-wise. This results in the following general definition:

$$\Phi_\varphi : E \dashrightarrow F \quad , \quad \Phi \subset F \boxplus \bar{E}$$
$$\varphi : M \longrightarrow N$$

Φ is a Dirac structure with support in graph(φ) that fits in the following short exact sequence:

$$0 \longrightarrow T^*\varphi \longrightarrow \Phi_\varphi \xrightarrow{\rho_E \oplus \rho_F} \mathrm{grph}(T\varphi) \longrightarrow 0$$

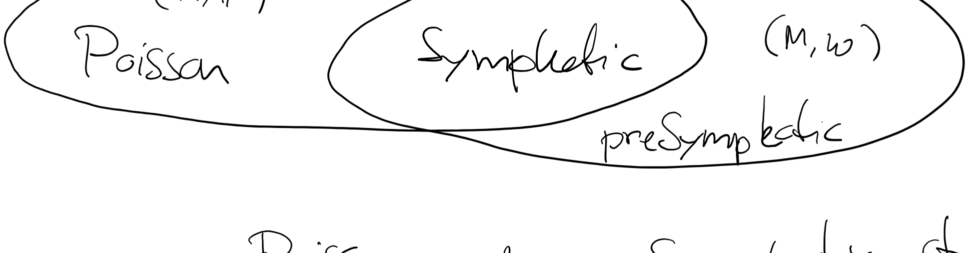
where $T^*\varphi$ denotes cotangent lift.

Composition of Courant morphisms may fail to be well-defined due to issues with cleanness of intersection.

IV RECOVERING KNOWN STRUCTURES FROM DIRAC GEOMETRY

The main advantage of the Courant algebroid formalism (and what motivated its development during the 1980s) is that it allows to encompass a huge breadth of geometric structures within a natural "category".

To illustrate this consider the following:



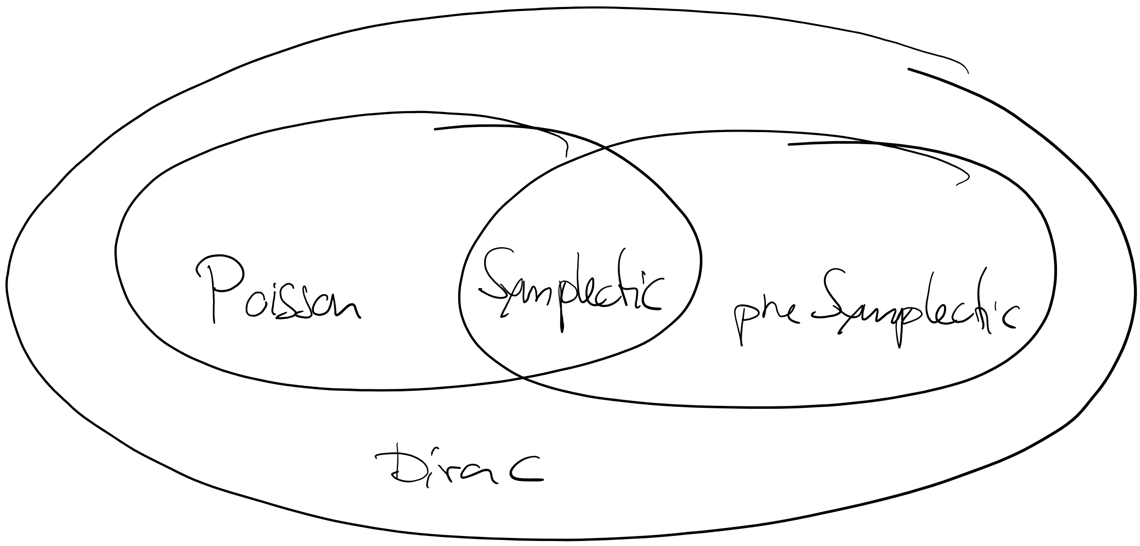
We can recover Poisson and preSymplectic structures on M trivially as Dirac structures on $\mathbb{T}M$:

Show

$\pi \in \mathfrak{X}^2(M)$ is Poisson $\Leftrightarrow$ $\mathrm{grph}(\pi^\#) \subset \mathbb{T}M$ is Dirac

$\omega \in \Omega^2(M)$ is Presymplectic $\Leftrightarrow$ $\mathrm{grph}(\omega^\flat) \subset \mathbb{T}M$ is Dirac

Furthermore, the introduction of 3-form backgrounds or "intermediate structures", i.e. ones not corresponding to graphs of tensors, are naturally regarded as Dirac structures:



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